

# TAIL EXPANSIONS FOR THE DISTRIBUTION OF THE MAXIMUM OF A RANDOM WALK WITH NEGATIVE DRIFT AND REGULARLY VARYING INCREMENTS

Ph. Barbe<sup>(1)</sup>, W.P. McCormick<sup>(2)</sup> and C. Zhang<sup>(2)</sup>

<sup>(1)</sup>CNRS, France, and <sup>(2)</sup>University of Georgia

**Abstract.** Let  $F$  be a distribution function with negative mean and regularly varying right tail. Under a mild smoothness condition we derive higher order asymptotic expansions for the tail distribution of the maxima of the random walk generated by  $F$ . An application to ruin probabilities is developed.

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**1. Introduction.** There is hardly a more basic stochastic model than a random walk, and for random walks with negative drift, a basic issue of study is the distribution of its global maximum. One reason for interest in this quantity is its connection to queueing processes. For a GI/G/1 queue, which is stable in the sense that the mean interarrival time exceeds the mean service time, the waiting time that the  $n$ -th arriving customer needs to wait until service begins has a limiting distribution as  $n$  tends to infinity. This is given by the distribution of the global maximum of a random walk with negative drift; see Asmussen (1987, §III.7). In an insurance-risk setting, the distribution of the global maximum of a random walk with negative drift directly appears in computing ruin probabilities over an infinite horizon; see, for example, Embrechts, Klüppelberg and Mikosch (1997, §1.1).

Let  $(X_n)_{n \geq 1}$  be a sequence of independent and identically distributed random variables having negative mean. The associated random walk is defined by  $S_0 = 0$  and for any integer  $n$  positive,  $S_n = X_1 + \dots + X_n$ . The distribution of its maximum,  $M = \max_{n \geq 0} S_n$ , can be represented as a compound-geometric distribution as follows. We first agree that the minimum of the empty set is  $+\infty$ . Then, let  $\tau$  denote the hitting time for the positive half-line

$$\tau = \min\{n : S_1 \leq 0, \dots, S_{n-1} \leq 0, S_n > 0\}.$$

This hitting time may be infinite, but it is finite with probability

$$p = P\{\tau < \infty\} = 1 - P\{S_1 \leq 0, S_2 \leq 0, \dots\}.$$

Recall that the first strict ascending ladder height distribution is defined by

$$F_+(x) = P\{S_\tau \leq x, \tau < \infty\}.$$

Since the random walk has a negative drift,  $F_+$  is a defective distribution with defect  $1 - p$ . By the Sparre Andersen identity (Feller, 1971, §XII.7) and Abel's lemma (Karlin, 1975, §II.5),

$$p = 1 - \exp\left(-\sum_{n \geq 1} \frac{1}{n} P\{S_n > 0\}\right);$$

see also Chung (1974). It follows that  $M$  has a compound-geometric distribution subordinate to the distribution  $H = p^{-1}F_+$  and with subordinator a geometric distribution with parameter  $p$ . More explicitly and following Feller (1971, §XII.5), writing  $H^{*n}$  the  $n$ -fold convolution of  $H$ , the distribution  $W$  of  $M$  is

$$W = (1 - p) \sum_{n \geq 0} p^n H^{*n}. \quad (1.1)$$

This step has replaced the original question of analyzing the distribution of global maximum of a random walk with the more elementary question of analyzing that of a compound sum, at the price, however, of introducing a derived distribution, namely, the ascending ladder height distribution, which requires its own analysis. In the case of a heavy-tailed step size distribution as prescribed by a subexponentiality assumption, Veraverbeke (1977) supplies an answer to this question through use of the distributional form of the Wiener-Hopf factorization. His result establishes inheritability of the subexponential property of the right Wiener-Hopf factor, i.e. that factor having mass concentrated in positive half line, from that of the underlying distribution. To state that result, we agree that for any possibly defective distribution function  $G$ , we write  $\overline{G}$  its tail, that is the function whose value at  $x$  is

$$\overline{G}(x) = \lim_{t \rightarrow \infty} G(t) - G(x).$$

Consider the Wiener-Hopf factorization  $F = F_+ + F_- - F_+ \star F_-$ , where  $F_-$  and  $F_+$  are concentrated on  $(-\infty, 0]$  and  $(0, \infty)$

respectively (see Feller, 1971, §XII.3). Let  $\mu$  be the mean of  $F$ , which we assume to be negative. Veraverbeke (1977) shows that, as  $x$  tends to infinity,

$$\overline{F}_+(x) \sim \frac{1-p}{-\mu} \int_x^\infty \overline{F}(t) dt.$$

We remark that the Wiener-Hopf factors are given by the strict ascending ladder height distribution and the weak descending ladder height distribution.

With these two steps in place, a first-order analysis of the distribution of  $M$  may be completed by using a result on tail-area asymptotics for subordinated probability distributions — in this case for the subordinator given by a geometric distribution. For example, the result in Athreya and Ney (1972, §IV.4) for first-order asymptotics of compound subexponential distributions with geometric subordinator gives the expected result that, under the assumption of  $F$  subexponential with negative mean  $\mu$ ,

$$\overline{W}(x) \sim \frac{1}{1-p} \overline{F}_+(x) \sim \frac{-1}{\mu} \int_x^\infty \overline{F}(t) dt,$$

as  $x$  tends to infinity. This last step of analysing tail areas for subordinated distributions is a subject of much interest in the literature. In the case of subexponential subordinate distributions, we mention the paper, Embrechts, Goldie and Veraverbeke (1979), who prove that for any integer valued random variable  $N$  independent of the sequence  $(X_i)_{i \geq 1}$  and such that  $Ez^N$  is analytic at  $z = 1$ ,

$$P\{S_N > x\} \sim EN P\{X_1 > x\}$$

as  $x$  tends to infinity.

Finally, we mention that when  $\overline{F}$  is regularly varying with index in the range from  $-1$  to  $-2$  and the mean is finite and negative, Omey and Willekens (1986) establish a second-order result for the tail  $\overline{W}$ ; see also Geluk (1992, 1996).

In a different range of tail heaviness which we will not explore in this paper, Feller (1971, §XII.5, Example c) considers the tail behavior of  $W$  when  $F$  has a moment generating function finite in a neighborhood of the origin. He shows that if the moment generating function of  $F$  is 1 at some positive  $\kappa$ , and if the number  $\beta = \int_0^\infty x \exp(\kappa x) dF_+(x)$  is finite, then

$$\overline{W}(x) \sim \frac{1-p}{\beta\kappa} e^{-\kappa x}$$

as  $x$  tends to infinity. If  $\beta$  is infinite, the result should be read as  $\overline{W} = o(e^{-\kappa x})$ .

**2. Expansions.** In this paper, we are interested in proving higher-order tail area asymptotics for the distribution  $W$  of the maximum  $M$  of the random walk. As previously noted, the representation of this distribution as a compound-geometric allows a restatement of this problem as one of establishing higher-order results for certain compound-geometric distributions. We remark that Omeij and Willekens (1987) provide second-order results for such a compound distribution subordinate to a distribution with finite first moment which satisfies certain smoothness and regularity conditions, e.g. membership in a subclass of subexponential distributions including distributions with regularly varying tails with index of regular variation at most  $-1$ . Note however that their results cannot be automatically applied to derive second-order behavior for  $M$ . This is because their result requires certain smoothness conditions which would need to be established for the right Wiener-Hopf factor under appropriate conditions on the underlying distribution.

Our approach to this question is to invoke a result from Barbe and McCormick (2004) on asymptotic expansions for tail areas of compound sums. Their Theorem 4.4.1 provides an  $m$ -term expansion for tail area, where the number of terms allowed is constrained by smoothness and moment conditions on the underlying distribution. To that end, we now present a smoothness condition needed for that result.

**Definition.** *A real measurable function  $f$  is smoothly varying with index  $-\alpha$  and order  $m$  if it is ultimately  $m$ -times continuously differentiable and the  $m$ -th derivative  $f^{(m)}$  is regular varying with index  $-\alpha - m$ . We denote the set of all such functions by  $SR_{-\alpha, m}$ .*

All distributions with regularly varying tails used in applications are smoothly varying of arbitrary order. Examples include the Pareto, Cauchy, Student, Burr and log-gamma distributions. Any function smoothly varying in the sense of Bingham, Goldie and Teugels (1984, §1.8.1) is smoothly varying of any fixed order.

The class  $SR_{-\alpha, m}$  may be extended to noninteger orders. This is useful to present sharp results. To define  $SR_{-\alpha, \omega}$  where  $\omega$  is a positive real number, we introduce the following notation. For any

function  $h$ , let

$$\Delta_{t,x}^r(h) = \text{sign}(x) \frac{h(t(1-x)) - h(t)}{|x|^r h(t)}.$$

**Definition.** Let  $\omega$  be a positive real number. Write  $\omega = m + r$  where  $m$  is the integer part of  $\omega$  and  $r$  is in  $[0, 1)$ . A function  $h$  is smoothly varying of index  $-\alpha$  and order  $\omega$  if it belongs to  $SR_{-\alpha,m}$  and

$$\lim_{\delta \rightarrow 0} \limsup_{t \rightarrow \infty} \sup_{0 < |x| \leq \delta} \Delta_{t,x}^r(h) = 0.$$

We write  $SR_{-\alpha,\omega}$  for the class of all such functions.

We remark that the spaces  $SR_{-\alpha,\omega}$  are nested, for  $SR_{-\alpha,r} \supset SR_{-\alpha,s}$  for  $r < s$ . In particular, if  $\omega$  is positive with integer part  $m$  and  $r = \omega - m$ , membership in  $SR_{-\alpha,\omega}$  is guaranteed by that in  $SR_{-\alpha,m+1}$ , that is by checking that the  $m+1$ -derivative is regularly varying of index  $-\alpha - m - 1$ . For further properties of smoothly varying functions of finite order, we refer to Barbe and McCormick (2004).

We now introduce the algebraic formalism to express the result. To that end let  $\mathbb{R}_m[D]$  denote the ring of real polynomials in  $D$  modulo the ideal generated by  $D^{m+1}$ . In other words, any polynomial in  $D$  divisible by  $D^{m+1}$  is set equal to 0.

For a possibly defective distribution function  $G$  with at least  $k$  moments finite, we write  $\mu_{G,k}$  its  $k$ -th moment. Note in particular that  $\mu_{G,0}$  is the total mass of  $G$ , equal to 1 if and only if  $G$  is not defective.

**Definition.** The Laplace character of order  $m$  of a possibly defective distribution  $G$  having a finite  $m$ -th moment is the element of  $\mathbb{R}_m[D]$  given by

$$L_{G,m} = \sum_{0 \leq k \leq m} \frac{(-1)^k}{k!} \mu_{G,k} D^k.$$

Note that since the map which associates to a measure its  $k$ -th moment is linear on its domain, the map  $G \mapsto L_{G,m}$  is linear on its domain.

The backward signed shift  $S$  on polymomials in  $D$  is defined linearly by  $SD^0 = 0$  and whenever  $j$  is a positive integer,  $SD^j = -D^{j-1}$ . It maps  $\mathbb{R}_m[D]$  to  $\mathbb{R}_{m-1}[D]$ .

We define the inverse of the differentiation on some functions as follows. If  $f$  is a function regularly varying of index less than  $-1$ , we set

$$D^{-1}f(t) = - \int_t^\infty f(x) dx.$$

Clearly,  $DD^{-1}$  is the identity on functions which are regularly varying of index less than  $-1$ , while  $D^{-1}D$  is the identity on the smoothly varying functions of negative index and order at least 1.

We now present our main result. Recall  $F$  denotes the step size distribution about which we assume its first moment is negative and that its right tail is regularly varying of index  $-\alpha$ . The strict ascending ladder height distribution is  $F_+$  and  $W$  is the distribution of  $M$ . Let

$$\kappa = \sup \left\{ r \geq 0 : \int_{-\infty}^0 |x|^r dF(x) < \infty \right\}.$$

This may be less than  $\alpha$  if the lower tail of  $F$  is heavier than the upper one. In the following theorem, it is implicitly supposed that  $\kappa$  is greater than 1.

**Theorem.** *Suppose that  $\overline{F}$  is smoothly varying of index  $-\alpha$  and order  $\omega$ . Then, for any integer  $m$  at least 1 and less than  $\omega \wedge \alpha \wedge \kappa$ , the moments  $\mu_{F_-,m}$  and  $\mu_{F_+,m-1}$  are finite and*

$$\overline{W} = (1-p)(\text{Id} - L_{F_+,m-1})^{-2}(SL_{F_-,m})^{-1}(D^{-1}\overline{F}) + o(\text{Id}^{-m+2}\overline{F}).$$

**Remark.** As previously mentionned, Laplace characters of order  $m-1$  are elements of the ring  $\mathbb{R}_{m-1}[D]$ . The inverses  $(\text{Id} - L_{F_+,m-1})^{-2}$  and  $(SL_{F_-,m})^{-1}$  are taken in that ring, multiplied together in that ring, and applied to  $D^{-1}\overline{F}$ .

**Remark.** The result may seem a little mysterious and not so explicit at a first glance. However, the computations related to Laplace characters can be implemented with a computer algebra package. For instance, the following very short **Maple** code calculates

the expansion given in the Theorem. In that code,  $F_p$  and  $F_m$  stand for  $F_+$  and  $F_-$ .

```
restart; m:=4: mu[Fp,0]:=1-q:
LFp:=sum('(-1)^j*mu[Fp,j]*x^j/j!', 'j'=0..m-1):
SLFm:=sum('(-1)^j*mu[Fm,j+1]*x^j/(j+1)!', 'j'=0..m-1):
a:=taylor((1-LFp)^(-2), x=0, m-1):
b:=taylor(SLFm^(-1), x=0, m-1):
expand(convert(q*taylor(a*b, x=0, m-1), polynom)/x);
```

One simply replaces  $x$  and  $q$  in the output by  $D$  and  $1 - p$ , with the convention that  $1/x$  should be replaced by  $D^{-1}$ . For instance, using that  $\mu_{F,1} = (1 - p)\mu_{F-,1}$ , taking  $m$  to be 4, we deduce the 3-terms expansion

$$\begin{aligned} \overline{W} &= \frac{1}{\mu_{F,1}} D^{-1} \overline{F} + \frac{1}{2\mu_{F,1}^2} ((1 - p)\mu_{F-,2} - 4\mu_{F+,1}\mu_{F-,1}) \overline{F} \\ &\quad + \frac{1}{12\mu_{F,1}^3} \left( 3(1 - p)^2 \mu_{F-,2}^2 + 12\mu_{F,1}(\mu_{F-,1}\mu_{F+,2} - \mu_{F+,1}\mu_{F-,2}) \right. \\ &\quad \left. + 36\mu_{F-,1}^2 \mu_{F+,1}^2 - 2(1 - p)\mu_{F,1}\mu_{F-,3} \right) \overline{F}' \\ &\quad + o(\text{Id}^{-1} \overline{F}). \end{aligned}$$

**Remark.** An important point to mention with regard to the main result is that the expansion it provides for the tail distribution  $\overline{W}$  is based on the underlying distribution  $F$  of the random walk, its derivatives and its integrated tail. This is notable, because the starting point to obtain this result is that of a tail area expansion for a subordinated distribution based on underlying distribution given by  $F_+$ . Since  $F_+$  is generally unattainable in an explicit form, the formulation of our main result is more attractive than that which results from a direct application of a tail area result for subordinated distributions. It is a comment on the usefulness of this algebraic approach that such an improvement is so easily and transparently attained compared to the effort to accomplish the same goal analytically. We conclude this remark by noting that our proof shows that a penultimate expansion based on  $F_+$  is given by

$$\overline{W} = (1 - p)(\text{Id} - L_{F+,m})^{-2} \overline{F}_+ + o(\text{Id}^{-m+1} \overline{F}) \quad (2.1)$$

provided  $m$  is less than  $\omega \wedge (\alpha - 1)$ . When  $m$  is 1 in the above, we obtain a second-order result in agreement with Theorem 2.2 in Omey

and Willekens (1987). Comparing this formula with that given in our theorem, we see that the latter is slightly less accurate. The reason is that, under the assumption of the theorem, replacement of  $\overline{F}_+$  with an approximation based on  $\overline{F}$ , its derivatives and its integral comes with a one-order lower error bound, viz.

$$\overline{F}_+ = (SL_{F_-,m})^{-1}D^{-1}\overline{F} + o(\text{Id}^{-m+2}\overline{F}).$$

**Application.** Finally, we present an application to insurance risk. To that end, we introduce some notation. Let  $R_0 = x$  be the initial capital of an insurance company. We assume that the claim amounts,  $(A_n)_{n \geq 1}$ , are independent, with common distribution function  $L$  having a smoothly varying tail of index  $-\alpha$  and order  $m+1$ . We also assume that the interclaim times  $(T_n)_{n \geq 1}$  are independent, with common distribution  $K$ , and independent of the claim amounts. Finally, we assume the intensity of the gross risk premium is some positive  $c$ . The net loss to the company in period  $n$  is  $X_n = A_n - cT_n$ . The sequence  $(X_n)_{n \geq 1}$  is a sequence of independent random variables with distribution  $F(x) = \int_0^\infty L(x+ct) dK(t)$ . Under the assumptions on  $L$ , it follows that  $F$  is ultimately  $m$ -times differentiable and

$$\frac{F^{(m)}(x)}{L^{(m)}(x)} = \int_0^\infty \frac{L^{(m)}(x+ct)}{L^{(m)}(x)} dK(t).$$

This implies that  $\overline{F}$  is smoothly varying of index  $-\alpha$  and order  $m$ . Let  $\psi(R_0)$  be the probability of eventual ruin given  $R_0$ . Writing as before  $S_n$  for the random walk with increment  $X_i$  and  $M$  for its maximum,  $\psi(x) = P\{M > x\}$ . We follow our established notation and set  $F_+$  and  $F_-$  for the strict ascending and weak descending ladder height distributions for the random walk  $S_n$ . We assume that  $X_1$  has negative expectation. We have following expansion of the ruin probability, which obviously follow from the theorem.

**Corollary.** *Assume that  $\overline{L}$  is smoothly varying of index  $-\alpha$  and order  $m+1$ . Assume also that  $\mu = EA_1 - cET_1$  is finite and negative and that  $m$  less than  $\alpha$ . Then,*

$$\begin{aligned} \psi(x) = (1-p)(\text{Id} - L_{F_+,m-1})^{-2}(SL_{F_-,m})^{-1}D^{-1}\overline{F}(x) \\ + o(x^{-m+2}\overline{F}(x)). \end{aligned}$$



**3. Proof of the theorem.** We first show that  $\mu_{F_-,m}$  is indeed finite. The distributional form of the Wiener-Hopf factorization implies that on the negative half-line,

$$F_- = F + F_+ \star F_- .$$

Since  $F_+$  has defect  $1-p$ , this yields  $F_- \leq F + pF_-$ , from which we deduce  $F_- \leq (1-p)^{-1}F$ . This proves the finiteness of  $\mu_{F_-,m}$ .

We begin the proof of the main part of our theorem by establishing a preparatory lemma.

**Lemma 1.** *Let  $(Y_i)_{i \geq 1}$  be a sequence of nonnegative random variables, independent and identically distributed with finite and positive mean. Let  $(Z_n)_{n \geq 0}$  be their corresponding random walk. Furthermore, let  $f$  be a regularly varying function of index  $-\beta$  less than  $-1$ . Then,*

$$\lim_{x \rightarrow \infty} \frac{1}{xf(x)} \sum_{n \geq 0} Ef(x + Z_n) = \frac{1}{(\beta - 1)EY_1} .$$

**Proof.** We may assume that  $f$  is ultimately positive. Since  $f$  is regularly varying of negative index, it is asymptotically equivalent to a nonincreasing function (Bingham, Goldie and Teugels, §1.5.2). Therefore, we can assume without any loss of generality that  $f$  is nonincreasing. Let  $\theta$  be positive and less than the mean of the  $Y_i$ 's. We have the trivial bound

$$\sum_{n \geq 0} Ef(x + Z_n) \leq \sum_{n \geq 0} f(x + \theta n) + f(x) \sum_{n \geq 0} P\{Z_n \leq \theta n\} .$$

Since  $f$  is regularly varying,

$$\sum_{n \geq 0} f(x + \theta n) \sim \int_0^\infty f(x + \theta s) ds .$$

as  $x$  tends to infinity. The change of variable  $\theta s = xz$  yields

$$\begin{aligned} \int_0^\infty f(x + \theta s) ds &= \frac{x}{\theta} \int_0^\infty f(x(1+z)) dz \\ &\sim \frac{xf(x)}{\theta} \int_0^\infty (1+z)^{-\beta} dz \\ &= \frac{xf(x)}{\theta(\beta-1)} . \end{aligned}$$

Let  $M$  be such that  $E(Y_1 \wedge M)$  is greater than  $\theta$ . Such  $M$  exists by monotone convergence of  $Y_1 \wedge M$  to  $Y_1$ . Let  $(Z_n^M)_{n \geq 0}$  be the random walk associated to the sequence  $(Y_i \wedge M)_{i \geq 1}$ . By the Hsu and Robbins theorem (see Chow and Teicher, 1988, §10.4), the series  $\sum_{n \geq 0} P\{Z_n^M \leq n\theta\}$  is finite. Since  $Z_n^M$  is at most  $Z_n$ , this series is at least  $\sum_{n \geq 0} P\{Z_n \leq n\theta\}$ , and the latter is finite as well. Therefore, since  $\theta$  is any positive number less than  $EY_1$ ,

$$\limsup_{x \rightarrow \infty} \frac{1}{xf(x)} \sum_{n \geq 0} Ef(x + Z_n) \leq \frac{1}{(\beta - 1)EY_1}.$$

To obtain a matching lower bound, let now  $\theta$  be a number greater than 1, and let  $\epsilon$  be a positive real number. Since  $f$  is ultimately positive and nonincreasing,  $\sum_{n \geq 0} Ef(x + Z_n)$  is ultimately at least

$$\sum_{n \geq 0} Ef(x + n(EY_1 + \epsilon)) \mathbb{I}\{|Z_n - nEY_1| \leq \epsilon n; \theta^{-1}x \leq nEY_1 \leq \theta x\}.$$

But if  $\theta^{-1}x \leq nEY_1 \leq \theta x$ , as  $x$  tends to infinity,

$$f(x + n(EY_1 + \epsilon)) \sim f(x) \left(1 + \frac{n}{x}(EY_1 + \epsilon)\right)^{-\beta}.$$

Moreover, in that range of  $n$ , for  $x$  large enough, the strong law of large numbers implies that  $P\{|Z_n - nEY_1| \leq n\epsilon\} \geq 1 - \epsilon$ . Therefore,  $\sum_{n \geq 0} Ef(x + Z_n)$  is ultimately at least

$$\begin{aligned} & (1 - \epsilon)f(x) \sum_{n \geq 0} \left(1 + \frac{n}{x}(EY_1 + \epsilon)\right)^{-\beta} \mathbb{I}\{\theta^{-1}x \leq nEY_1 \leq \theta x\} \\ & \sim (1 - \epsilon)f(x) \int_{\theta^{-1}x/EY_1}^{\theta x/EY_1} \left(1 + \frac{s}{x}(EY_1 + \epsilon)\right)^{-\beta} ds \\ & = (1 - \epsilon)f(x) \frac{1}{1 - \beta} \frac{x}{EY_1 + \epsilon} \left[ \left(1 + \frac{s}{x}(EY_1 + \epsilon)\right)^{-\beta+1} \right]_{\theta^{-1}x/EY_1}^{\theta x/EY_1}. \end{aligned}$$

Since  $\theta$  and  $\epsilon$  are arbitrary, we can make  $\theta$  tend to infinity after taking the asymptotic equivalent of the lower bound as  $x$  tends to infinity, proving that

$$\sum_{n \geq 0} Ef(x + Z_n) \geq (1 + o(1)) \frac{xf(x)}{EY_1(\beta - 1)}$$

as  $x$  tends to infinity. ■

Note that with unimportant and additional assumptions an alternate proof of Lemma 1 based on the renewal theorem may be given, slightly shorter, but not as direct. To sketch it, write  $G$  the distribution function of  $Y_i$  and consider the renewal function  $U = \sum_{n \geq 0} G^{*n}$ . We see that  $\sum_{n \geq 0} Ef(x + Z_n) = \int f dU$ . When  $f$  is smooth, an integration by parts and a change of variable bring this integral to the form  $s \int_1^\infty f'(xs)U(x(s-1)) ds$ . The renewal theorem (Feller, 1971, §XI.3) yields  $U(x(s-1)) \sim x(s-1)/EY_1$  as  $x$  tends to infinity, uniformly in  $s$  at least 1. The result then follows by standard arguments involving regular variation.

The main argument for proving our theorem is to show that  $\overline{F}_+$  is smoothly varying of index  $-\alpha + 1$  and same order as  $\overline{F}$ . This is stated in the next lemma.

**Lemma 2.** *The strict ascending ladder height distribution  $F_+$  is smoothly varying of index  $-\alpha + 1$  and same order  $\omega$  as  $F$ . Moreover,*

$$\overline{F}_+ \sim \frac{-1}{(\alpha - 1)\mu_{F_-,1}} \text{Id } \overline{F}.$$

**Proof.** The proof has four steps.

*Step 1.* A representation for  $\overline{F}_+$ . By the distributional form of Wiener-Hopf factorization, we have

$$F = F_+ + F_- - F_+ \star F_- . \quad (3.1)$$

It is convenient to introduce the following integral operator,

$$U_{F_-} g(t) = \int_{-\infty}^0 g(t-u) dF_-(u) .$$

As usual, powers of operators are defined inductively. In particular,  $U_{F_-}^0$  is the identity and  $U_{F_-}^n = U_{F_-} \circ U_{F_-}^{n-1}$  for any integer  $n$  positive. On  $(0, \infty)$ , we can write (3.1) as  $\overline{F}_+ = \overline{F} + \overline{F_+ \star F_-}$ , which leads to

$$\overline{F}_+ = \overline{F} + U_{F_-} \overline{F}_+ . \quad (3.2)$$

By recursion this yields

$$\overline{F}_+ = \sum_{0 \leq i \leq n} U_{F_-}^i \overline{F} + U_{F_-}^{n+1} \overline{F}_+ .$$

Note that  $F_-$  cannot be the distribution degenerate at 0 since  $F$  is assumed to have a negative mean. Let  $(Y_i)_{i \geq 1}$  be a sequence of independent random variables, all with the same distribution  $F_-$ , and let  $(Z_n)_{n \geq 0}$  be their random walk (note that the signs are changed compared to the previous lemma). Observe that  $\sum_{0 \leq i \leq n} U_{F_-}^i \bar{F}$  is nondecreasing in  $n$  and that, by dominated convergence,  $U_{F_-}^{n+1} \bar{F}_+(x) = E \bar{F}_+(x - Z_{n+1})$  tends to 0 as  $n$  goes to infinity. Consequently, we obtain the representation

$$\bar{F}_+ = \sum_{i \geq 0} U_{F_-}^i \bar{F}. \quad (3.3)$$

Note that combined with Lemma 1, this representation yields Veraverbeke's (1977) theorem asserting that

$$\bar{F}_+ \sim \frac{-1}{(\alpha - 1)\mu_{F_-,1}} \text{Id } \bar{F}. \quad (3.4)$$

*Step 2. A representation for  $\bar{F}_+^{(k)}$ .* Let  $k$  be a positive integer at most  $\omega \wedge (\alpha - 1)$ . Using the mean value theorem, there exists a sequence of real numbers,  $(\theta_n)_{n \geq 0}$ , nonnegative and at most 1, such that

$$\begin{aligned} \sum_{n \geq 0} \left| \frac{1}{\epsilon} \left( U_{F_-}^n \bar{F}^{(k-1)}(x + \epsilon) - U_{F_-}^n \bar{F}^{(k-1)}(x) \right) - U_{F_-}^n \bar{F}^{(k)}(x) \right| \\ = \sum_{n \geq 0} |U_{F_-}^n \bar{F}^{(k)}(x + \theta_n \epsilon) - U_{F_-}^n \bar{F}^{(k)}(x)| \\ \leq \sum_{n \geq 0} E |\bar{F}^{(k)}(x + \theta_n \epsilon - Z_n) - \bar{F}^{(k)}(x - Z_n)|. \end{aligned}$$

Since the absolute value of a difference is at most the sum of the absolute values, Lemma 1 shows that the above series is bounded as a function of  $x$  and uniformly in  $\epsilon$  in some interval  $(0, \eta)$ . Moreover, every summand tends to 0 as  $\epsilon$  tends to 0. Therefore, the series tends to 0 as  $\epsilon$  tends to infinity. This proves that

$$\bar{F}_+^{(k)} = \sum_{i \geq 0} U_{F_-}^i \bar{F}^{(k)} \quad (3.5)$$

on some neighborhood of infinity.

*Step 3.*  $\overline{F}_+^{(k)}$  is regularly varying. The asymptotic equivalence in (3.4) implies that  $\overline{F}_+$  is regularly varying with index  $-\alpha + 1$ . Recall that  $k$  is at most  $\omega \wedge (\alpha - 1)$ . By assumption  $\overline{F}^{(k)}$  is regularly varying. By representation (3.5) and Lemma 1,  $\overline{F}_+^{(k)}$  is regularly varying of index  $-\alpha - k + 1$ . Taking  $k$  to be  $m$ , that is  $\lfloor \omega \rfloor$ , this proves that  $\overline{F}_+$  is smoothly varying of index  $-\alpha + 1$  and order  $m$ .

*Step 4. Concluding the proof of the lemma.* Following Barbe and McCormick (2004), for a function  $h$  define

$$\overline{\Delta}_{\tau, \delta}^r(h) = \sup_{t \geq \tau} \sup_{0 < |x| \leq \delta} |\Delta_{t, x}^r h|.$$

This quantity is nonincreasing in  $\tau$  and nondecreasing in  $\delta$ . Using representation (3.5), we see that

$$\begin{aligned} \overline{F}_+^{(m)}(t(1-x)) - \overline{F}_+^{(m)}(t) \\ = \sum_{n \geq 0} E \left( F^{(m)}(t(1-x) - Z_n) - F^{(m)}(t - Z_n) \right). \end{aligned}$$

Consider  $x$  in the range  $[-\delta, \delta] \setminus \{0\}$ . Factoring  $t - Z_n$  in  $t(1-x) - Z_n$ , the  $n$ -th summand in the series above is at most

$$\left( \frac{t}{t - Z_n} |x| \right)^r |\overline{F}^{(m)}(t - Z_n)| \overline{\Delta}_{t - Z_n, t\delta/(t - Z_n)}^r \overline{F}^{(m)}.$$

Consequently, for  $|x|$  positive and at most  $\delta$ ,

$$|\Delta_{t, x}^r \overline{F}_+^{(m)}| \leq E \sum_{n \geq 0} \left| \frac{\overline{F}^{(m)}(t - Z_n)}{\overline{F}_+^{(m)}(t)} \right| \overline{\Delta}_{t, \delta}^r \overline{F}^{(m)}.$$

It follows from step 3,  $\overline{F}_+^{(m)} \asymp \text{Id} \overline{F}^{(m)}$ . It then follows from Lemma 1 and our assumption on  $F$  that

$$\lim_{\delta \rightarrow 0} \lim_{t \rightarrow \infty} \sup_{0 < |x| < \delta} |\Delta_{t, x}^r \overline{F}_+^{(m)}| = 0,$$

proving the smooth variation of order  $\omega$  of  $\overline{F}_+$ . ■

Finally, we present a technical lemma of some independent interest, particularly in the light of Marić's (2000) work. It is needed for the proof of our main result. We remark that the result is not proved under optimal conditions.

**Lemma 3.** *Let  $(a_i)_{0 \leq i \leq m}$  be a sequence of real numbers with  $a_0$  different from 0. For any nonnegative integer  $k$  at most  $m$ , define the differential operators  $P_k(D) = \sum_{0 \leq i \leq k} a_i D^i$ . Let  $\psi$  be a function. Let  $f$  and  $g$  be two functions smoothly varying with index  $-\alpha$  and order at least  $m$  satisfying the differential equations*

$$P_{m-k}(D)D^k f = D^k g + o(\psi), \quad k = 0, 1, \dots, m.$$

*Then, viewing  $P_m(D)$  in  $\mathbb{R}_m[D]$ ,*

$$f = P_m(D)^{-1}g + o(\psi).$$

The lemma may be interpreted as saying that if the functions  $D^k g$  have a generalized asymptotic expansion in the asymptotic scale  $D^k f$ , then  $f$  has a generalized asymptotic expansion in the asymptotic scale  $D^k g$ .

**Proof.** Write  $b_k$  the  $k$ -th coefficient of  $P_m(D)^{-1}$ . Then

$$P_m(D)^{-1}g = \sum_{0 \leq k \leq m} b_k D^k g. \quad (3.6)$$

In this sum, by assumption, we can replace  $D^k g$  by  $P_{m-k}(D)D^k f + o(\psi)$ . Since  $P_{m-k}(D)D^k = P_m(D)D^k$  in  $\mathbb{R}_m[D]$ , the definition of the  $b_k$  and (3.6) yield  $P_m(D)^{-1}g = f + o(\psi)$ , which is the result. ■

We now conclude the proof of the main theorem. Using (3.2) and applying Lemma 2 and a variant of Theorem 2.3.1 in Barbe and McCormick (2004), we obtain for any nonnegative  $k$  at most  $m$ ,

$$\begin{aligned} \overline{F}^{(k)}_+ &= \overline{F}^{(k)}_+ - L_{F_-, m-k} \overline{F}^{(k)}_+ + o(\text{Id}^{-m} \overline{F}_+) \\ &= SL_{F_-, m-k} D^{k+1} \overline{F}_+ + o(\text{Id}^{-m} \overline{F}_+). \end{aligned}$$

By Veraverbeke's (1977) theorem or (3.4), this implies

$$SL_{F_-, m-k} D^k D \overline{F}_+ = D^k \overline{F}_+ + o(\text{Id}^{-m+1} \overline{F}_+).$$

Applying Lemma 3, we obtain

$$D \overline{F}_+ = (SL_{F_-, m})^{-1} \overline{F}_+ + o(\text{Id}^{-m+1} \overline{F}_+).$$

Hence, integrating,

$$\overline{F}_+ = (SL_{F_-,m})^{-1}D^{-1}\overline{F} + o(\text{Id}^{-m+2}\overline{F}). \quad (3.7)$$

Representation (1.1) and Theorem 4.4.1 in Barbe and McCormick (2004), upon noting that if  $N$  is a random variable with geometric distribution with parameter  $p$ , then  $ENL_{H,m}^{N-1}$  is  $p(1-p)(\text{Id} - pL_{H,m})^{-2}$  in  $\mathbb{R}_m[D]$ , yield formula (2.1), that is

$$\overline{W} = (1-p)(\text{Id} - L_{F_+,m})^{-2}\overline{F}_+ + o(\text{Id}^{-m}\overline{F}_+).$$

To obtain the statement of the Theorem, we again use representation (1.1) and apply Theorem 4.4.1 in Barbe and McCormick (2004) to obtain that if  $m$  is less than  $\alpha \wedge \omega$ ,

$$\overline{W} = (1-p)(\text{Id} - L_{F_+,m-1})^{-2}\overline{F}_+ + o(\text{Id}^{-m+1}\overline{F}_+).$$

Then, we use Lemma 2 and (3.7) to conclude. ■

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Ph. Barbe  
90 rue de Vaugirard  
75006 PARIS  
FRANCE

W.P. McCormick and C. Zhang  
Dept. of Statistics  
University of Georgia  
Athens, GA 30602  
USA  
{bill,czhang}@stat.uga.edu